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Existence of localised solutions of (1 + 1)-dimensional non-linear Dirac equations with scalar self-interaction

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Abstract. Necessary and sufficient conditions for the existence of localised solutions of the form $\psi(x, t) = \exp(-i\omega t)\phi(x)$, with ϕ real, of the classical equations of motion for $(1+1)$-dimensional non-linear spinor fields are presented. Furthermore, we give existence conditions for non-linear second-order equations obtained as a Klein-Gordon limit of the considered spinor field equations.

1. Introduction

Much progress has been made in the mathematical treatment of classical non-linear Dirac fields. Necessary conditions for the existence of stationary solutions have been obtained in arbitrary space dimensions [1, 2]. Furthermore, two recent papers present explicit solutions of non-linear Dirac equations in one space dimension which give important hints for general existence conditions [3, 4]. There are no analytic solutions in three space dimensions and most understanding is based on numerical investigations. Only for the scalar self-interactions is there a rigorous proof, by Cazenave and Vázquez, for the existence of stationary solutions, under simple assumptions on the self-interaction term which are explained below [5]. Using the 'shooting method' they show the existence of solutions which are separable in spherical coordinates, a special feature of the scalar self-interaction.

In this paper we restrict ourselves to the scalar self-interaction for Dirac equations in $(1+1)$ dimensions. Thus we consider the Lagrangian

$$L_D = \tfrac{1}{2}i[\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi] - m\psi\bar{\psi} + G(\psi\bar{\psi}). \qquad (1.1)$$

Looking for stationary states, i.e.

$$\psi(x, t) = \exp(-i\omega t)\phi(x) \qquad \phi(x) = \begin{pmatrix} v(x) \\ u(x) \end{pmatrix} \qquad (1.2)$$

of the associated field equation and requiring ϕ to be real, we are led to the following system of ordinary differential equations:

$$\begin{aligned} u' &= v[g(v^2 - u^2) - (m - \omega)] \\ v' &= u[g(v^2 - u^2) - (m + \omega)] \end{aligned} \qquad (1.3)$$

with positive constants m and ω and where g denotes the derivative of G. Since u and v have to vanish at infinity we have to consider a boundary value problem.

The existence condition of [2] is

$$\int_{\mathbb{R}} (v^2-u^2)g(v^2-u^2)-G(v^2-u^2)\,\mathrm{d}x>0 \tag{1.4}$$

if (u, v) is a localised solution of (1.3). Furthermore, we obtain that

$$\omega\int_{\mathbb{R}} uv\,\mathrm{d}x=0 \tag{1.5}$$

and

$$H(u(x), v(x))=\tfrac{1}{2}[G(v^2(x)-u^2(x))-m(v^2(x)-u^2(x))+\omega(v^2(x)+u^2(x))]=0. \tag{1.6}$$

In § 2 of this paper, under certain assumptions on g, we give a necessary and sufficient condition for the existence of a localised solution. Furthermore, the obtained solution exhibits certain nice properties, e.g. exponential decay. In § 3 we consider a non-linear scalar field equation in one space dimension obtained as a Klein-Gordon limit of the spinor field equation in § 2. Applying the existence theorem of Berestycki and Lions [6] we obtain a condition which is both necessary and sufficient for the existence of a localised solution.

2. An existence theorem for localised solutions of non-linear Dirac equations

Let $g\in C(\mathbb{R},\mathbb{R})$ be a locally Lipschitz continuous increasing function such that $g(0)=0$.

Using $G(\zeta)=\int_0^\zeta g(s)\,\mathrm{d}s$ we define

$$H_0(\zeta)\equiv H(0,\zeta)=\tfrac{1}{2}[G(\zeta^2)-m\zeta^2+\omega\zeta^2]. \tag{2.1}$$

We consider the boundary value problem

$$\begin{aligned}
&u'=v[g(v^2-u^2)-(m-\omega)]\\
&v'=u[g(v^2-u^2)-(m+\omega)]\\
&u, v\in C^1(\mathbb{R},\mathbb{R})\\
&\lim_{x\to\pm\infty} v(x)=\lim_{x\to\pm\infty} u(x)=0\\
&v(x_0)>0 \qquad u(x_0)=0 \qquad \text{for some } x_0\in\mathbb{R}.
\end{aligned} \tag{2.2}$$

Theorem 2.1. A necessary and sufficient condition for the existence of a solution (u, v) of problem (2.2) is that

$$\begin{aligned}
&\zeta_0=\inf\{\zeta>0\,|\,H_0(\zeta)=0\} \text{ exists}\\
&\zeta_0>0 \qquad \mathrm{d}H_0/\mathrm{d}\zeta|_{\zeta=\zeta_0}>0.
\end{aligned} \tag{2.3}$$

Furthermore, if (2.3) is satisfied, the solution is unique up to translations of the origin and satisfies, after suitable translations of the origin,

(i) $-u(-x)=u(x)$ $\qquad v(-x)=v(x)$

(ii) $v(0)=\zeta_0$ $\qquad u(0)=0$

(iii) $0<u(x)<v(x)$ $\qquad$ for $x>0$

(iv) $u(x)$, $v(x)$ have exponential decay at infinity.

Remark 2.2. Under the assumptions of theorem 2.1 we can obtain the solution as the solution of the initial value problem

$$\begin{aligned} u' &= v[g(v^2-u^2)-(m-\omega)] \\ v' &= u[g(v^2-u^2)-(m+\omega)] \\ u(0) &= 0 \qquad v(0)=\zeta_0. \end{aligned} \tag{2.4}$$

Remark 2.3. Condition (2.3) implies $H_0(\zeta)<0$ in $(0,\zeta_0)$ and $m>\omega$ since $\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta|_{\zeta=0}\leq 0$.

Remark 2.4. Our assumptions on g are very similar to the assumptions on g for the existence theorem in three space dimensions but they are more general (see [5]).

Proof of theorem 2.1. Let (u, v) denote the solution of the initial value problem (2.4). It exists and is unique on a maximal interval $(-\bar{x}, \bar{x})$. Furthermore, $H(u(x), v(x))=0$ for $|x|<\bar{x}$. Since

$$\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta|_{\zeta=\zeta_0}>0$$

we have $u'(0)>0$ while $v'(0)=0$. We observe

$$\begin{aligned} -u(-x) &= u(x) \qquad |x|<\bar{x} \\ v(-x) &= v(x) \qquad |x|<\bar{x}. \end{aligned}$$

Thus we consider only the interval $(0, \bar{x})$.

We claim that $u(x)>0$, $v(x)>0$ in $(0,\bar{x})$, from the following.

(i) Suppose there exists $x_0>0$ with $u(x_0)=0$, $v(x_0)>0$ ($v(x_0)\leq 0$ is impossible). Then $v(x_0)\geq \zeta_0$ because of the definition of ζ_0. But $\delta(x)=v^2(x)-u^2(x)$ is decreasing in $(0, x_0)$ which yields a contradiction. Therefore $u(x_0)>0$ in $(0,\bar{x})$.

(ii) Suppose there exists x_1 with $v(x_1)=0$. If $u(x_1)>0$ the solutions intersect in $(0, x_1)$ which is impossible because of $H(u(x), v(x))=0$. If $u(x_1)<0$ there exists $x_0>0$ with $u(x_0)=0$ which is ruled out in (i). If $u(x_1)=0$ then $v'(x_1)=u'(x_1)=0$. Thus $u=v\equiv 0$, which is impossible. Therefore $v(x)>0$ in $(0,\bar{x})$.

Furthermore, clearly $0<u(x)<v(x)$ in $(0,\bar{x})$.

Next we show that (u, v) is bounded in $(0,\bar{x})$. We have $\delta(x)>0$ in $(0,\bar{x})$ and $H_0(\delta(x))=-\omega u^2(x)<0$. Since $H_0(\zeta)$ is bounded below, $u(x)$ is bounded. Using $\delta(x)<\zeta_0$ in $(0,\bar{x})$ we see that $v(x)$ is also bounded. By continuation arguments one can see that (u, v) is defined on the whole line, i.e. $\bar{x}=\infty$.

At last $\lim_{x\to\infty} v(x)=\lim_{x\to\infty} u(x)=0$. Indeed, we have $\lim_{x\to\infty} u'(x)=\lim_{x\to\infty} v'(x)=0$ because (u, v) is bounded and (u, v) cannot oscillate. Clearly we have $\lim_{x\to\infty} u(x)=0$. Now let $\lim_{x\to\infty} v(x)=v_\infty$. Then $H_0(v_\infty)=0$ and $\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta|_{\zeta=v_\infty}=0$. Thus $v_\infty\neq\zeta_0$. But $v_\infty>\zeta_0$ is impossible and therefore $v_\infty=0$. The proof of the exponential decay is now immediate.

Now suppose that (w, z) is another solution of (2.2). After a suitable translation of the origin we have $w(0)=0$, $z(0)>0$ and $H_0(z(0))=0$. Thus $z(0)=\zeta_0$ and $u\equiv w$, $v\equiv z$ by the uniqueness of the initial value problem (2.4).

We claim that condition (2.3) is necessary. Suppose it is violated and there exists a solution (u, v) of (2.2). Clearly, after translation $H_0(v(0)=0$. Hence ζ_0 exists and (2.3) can only fail to be satisfied in two ways, as follows.

Case 1. $\zeta_0 > 0$ but $\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta|_{\zeta=\zeta_0} \leqslant 0$.

Suppose $v(0) = \zeta_0$. Then $\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta|_{\zeta=\zeta_0} < 0$ because otherwise $(v(0), 0)$ is a rest point of $H(u, v)$. As a consequence $H_0(\zeta) \geqslant 0$ in $(0, \zeta_0)$ and $\omega \geqslant m$ because $\mathrm{d}H_0(\zeta)/\mathrm{d}\zeta \geqslant 0$ for small ζ. But then we have $g(\zeta_0^2) < m - \omega \leqslant 0$ which contradicts the assumptions on g. If $v(0) > \zeta_0$ there exists $x_0 > 0$ such that $v(x_0) = \zeta_0$ because (u, v) solves the boundary value problem. We have $0 < \delta(x_0) \leqslant \zeta_0$ and $H_0(\delta(x_0)) = -\omega u^2(x_0) \geqslant 0$ which implies $u(x_0) = 0$ and we conclude as above. Hence this case is ruled out.

Case 2. $\zeta_0 = 0$.

Here $H_0(\zeta) > 0$ for $\zeta > 0$ and there exists no solution of (2.2).

A nice property of the upper spinor component v is given in the following.

Corollary 2.5. Under the assumptions of theorem 2.1 the upper spinor component v is decreasing in $\mathbb{R}^+$ if and only if

$$2\omega\zeta_0 \geqslant \left.\frac{\mathrm{d}H_0(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta=\zeta_0} (>0). \tag{2.5}$$

Proof. (*a*) If $g(\zeta_0^2) - (m + \omega) < 0$, $v(x)$ is decreasing for small x. Suppose there exists x_0 with $v'(x_0) = 0$. Then $g(\delta(x_0)) - (m + \omega) = 0$ because $u(x) > 0$ in $\mathbb{R}^+$. But this is impossible because $\delta(x_0) < \zeta_0$ and g is increasing.

If $g(\zeta_0^2) - (m + \omega) = 0$ we see that $g(\delta(x)) - (m + \omega) < 0$ because $\delta(x)$ is decreasing for small x. Thus $v(x)$ decreases for small x and we conclude as before.

(*b*) If $g(\zeta_0^2) - (m + \omega) > 0$, $v(x)$ increases for small x.

Thus we have proved the corollary.

Remark 2.6. In order to compare the existence condition (2.3) with the integral condition (1.4) we observe that (2.3) implies

$$\zeta_0^2 g(\zeta_0^2) > G(\zeta_0^2). \tag{2.6}$$

(2.6) can be understood as a local condition for the existence of a solution, or more precisely a 'one-point' condition. Since $\delta(x)$ must be non-negative we have $\delta(x)g(\delta(x)) > G(\delta(x))$ for all $x \in \mathbb{R}$ by the properties of g.

Remark 2.7. It should be stressed that in the case of one space dimension existence of a solution also implies uniqueness. We do not know which possibilities occur in the multidimensional case.

3. Existence conditions for localised solutions of Klein–Gordon-type equations

If one considers the Klein-Gordon limit associated with the spinor field equation of (1.1) we are led to the following non-linear scalar field equation when looking for stationary solutions (see [1, 4]):

$$-v'' = -(m^2 - \omega^2)v + 2mg(v^2)v - g(v^2)g(v^2)v. \tag{3.1}$$

We seek solutions which vanish at infinity and for which $v(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Denoting the right-hand side of (3.1) by $f(v)$ and defining $F(v) = \int_0^v f(s)\,\mathrm{d}s$ we have the well known existence theorem of Berestycki and Lions (see [6]), as follows.

Theorem 3.1. A necessary and sufficient condition for the existence of a solution v of equation (3.1) with the above boundary conditions is that

$$\begin{aligned} &\zeta_0 = \inf\{\zeta > 0 \mid F(\zeta) = 0\} \text{ exists} \\ &\zeta_0 > 0 \qquad f(\zeta_0) > 0. \end{aligned} \tag{3.2}$$

Furthermore, if (3.2) is satisfied, then (3.1) with boundary conditions has a unique solution up to translations of the origin and this solution satisfies (after a suitable translation of the origin):

(i) $v(-x) = v(x) \qquad x \in \mathbb{R}$

(ii) $v(x) > 0 \qquad x \in \mathbb{R}$

(iii) $v(0) = \zeta_0$

(iv) $v'(x) < 0 \qquad x \in \mathbb{R}^+$

(v) $v(x)$ has exponential decay at infinity.

Now we want to investigate when condition (3.2) is satisfied. For this purpose we make an additional assumption on g:

$$\lim_{t \to \infty} g(t) > m + \omega.$$

Let us remark that this assumption was also made in [5] when proving the existence of localised solutions for the non-linear Dirac equation in three space dimensions.

We now state the following proposition.

Proposition 3.2 Condition (3.2) holds if and only if

$$\int_0^{m+\omega} g^{-1}(\sigma)(\sigma - m)\, \mathrm{d}\sigma > 0$$

where g^{-1} denotes the inverse of g.

Proof. We have

$$\begin{aligned} F(v) &= -\tfrac{1}{2}(m^2 - \omega^2)v^2 + 2m \int_0^v g(s^2)s\, \mathrm{d}s - \int_0^v g(s^2)g(s^2)s\, \mathrm{d}s \\ &= -\tfrac{1}{2}(m^2 - \omega^2)v^2 + m \int_0^{v^2} g(t)\, \mathrm{d}t - \tfrac{1}{2} \int_0^{v^2} g(t)^2\, \mathrm{d}t. \end{aligned}$$

It is easy to see that F has three critical points on $\mathbb{R}_0^+$, namely $v_0 = 0$, $v_1 = (g^{-1}(m - \omega))^{1/2}$ and $v_2 = (g^{-1}(m + \omega))^{1/2}$. In v_1 there is a local minimum of F while v_2 is the place of the local maximum. Now condition (3.2) is satisfied if and only if $F(v_2) > 0$.

By the substitution $\sigma = g(t)$ we obtain

$$\begin{aligned} F(v_2) &= -\tfrac{1}{2}(m^2 - \omega^2)g^{-1}(m + \omega) + m \int_0^{m+\omega} \sigma \frac{\mathrm{d}g^{-1}(\sigma)}{\mathrm{d}\sigma}\mathrm{d}\sigma - \tfrac{1}{2} \int_0^{m+\omega} \sigma^2 \frac{\mathrm{d}g^{-1}(\sigma)}{\mathrm{d}\sigma}\, \mathrm{d}\sigma \\ &= -\tfrac{1}{2}(m^2 - \omega^2)g^{-1}(m + \omega) + m(m + \omega)g^{-1}(m + \omega) \\ &\quad - m \int_0^{m+\omega} g^{-1}(\sigma)\, \mathrm{d}\sigma - \tfrac{1}{2}(m + \omega)^2 g^{-1}(m + \omega) \\ &\quad + \int_0^{m+\omega} \sigma g^{-1}(\sigma)\, \mathrm{d}\sigma = \int_0^{m+\omega} g^{-1}(\sigma)(\sigma - m)\, \mathrm{d}\sigma \end{aligned}$$

which proves the proposition.

Remark 3.3. The existence condition of proposition 3.2 also holds for the Klein-Gordon limit of Dirac equations with scalar self-interaction in three space dimensions. Even in that case this existence condition is both necessary and sufficient.

Remark 3.4. A consequence of proposition 3.2 is the following. There exists $\omega_c \in (0, m)$ such that the associated Klein-Gordon equation has no localised solution if $\omega \leqslant \omega_c$ and has a solution obtained by theorem 3.1 (resp the existence theorem in the multi-dimensional case, see [6]) if $\omega > \omega_c$.

A first result in this direction was obtained by Vázquez [1] who determined ω_c for the Klein-Gordon limit of the Fermi interaction $\lambda\bar{\psi}\psi$.

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